Lecture 15

7.8 - Improper Integrals

In this section, we extend the concept of an integral to infinite lengths and to functions with an infinite discontinuity in the interval of integration.

Examples of these:

$$\int_1^\infty \frac{1}{x^3}\,dx, \quad \int_0^1 \frac{1}{x^3}\,dx, \quad \int_{-\infty}^\infty \frac{1}{4+x^2}\,dx$$

Let's deal first with infinite intervals:

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided the integral exists and is finite.

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided the limit exists and is finite.

Terminology: The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called convergent if the corresponding limits exist and are finite, and are called divergent otherwise.

(c) If (for any value of c) both $\int_{-\infty}^{c} f(x)\,dx$ & $\int_{c}^{\infty} f(x)\,dx$ are convergent, then we define:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

Ex: Determine whether the following integrals converge or diverge:

ⓐ $\int_{1}^{\infty} \frac{1}{x}\,dx$ ⓑ $\int_{-\infty}^{5} e^{2x}\,dx$

<u>Ex</u>: Determine whether $\int_{-\infty}^{\infty} \frac{1}{4+x^2}\,dx$ is convergent or divergent. If it is convergent, find its value.

<u>Theorem</u>: $\int_{1}^{\infty} \frac{1}{x^p}\,dx$ is convergent if $p>1$, divergent if $p\le 1$

The next kind of improper integral is when the integrand has an infinite discontinuity in the interval of integration. This again has 3 cases, all of which are handled in the same way as before:

ⓐ If f is continuous on $[a,b)$ and is discontinuous at b, then $$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$ if the limit exists and is finite.

ⓑ If f is continuous on $(a,b]$ and is discontinuous at a, then $$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$ if the limit exists and is finite.

Once again, we call the improper integral $\int_a^b f(x)\,dx$ is called convergent if the corresponding limit exists and is finite, and is called divergent otherwise.

ⓒ If f has a discontinuity at c, where $a<c<b$, and both $\int_a^c f(x)\,dx$ & $\int_c^b f(x)\,dx$ are convergent, then we define $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example Determine whether $\int_0^2 \frac{1}{x-2}\,dx$ is convergent or divergent.

Theorem $\int_0^1 \frac{1}{x^p}\,dx$ is divergent if $p \geq 1$ & convergent if $p<1$.

(Notice this is backwards to before!)

Ex: Determine whether $\int_{-2}^{3} \frac{1}{(x-1)^4}\,dx$ is convergent or divergent. If it is convergent, give its value

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Comparison Test for Integrals

If f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$, then

ⓐ If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent

ⓑ If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Ex: Use the comparison test to determine whether the following integrals are convergent or divergent.

ⓐ $\displaystyle\int_1^\infty \frac{1}{x^2+x+1}\,dx$ ⓑ $\displaystyle\int_1^\infty \frac{1}{x-\frac{1}{2}}\,dx$ ⓒ $\displaystyle\int_0^\pi \frac{\cos^2 x}{\sqrt{x}}\,dx$

ⓓ $\displaystyle\int_0^\infty \frac{e^{-x}}{1+\sin^2 x}\,dx$